

Middlesex University Research Repository

An open access repository of

Middlesex University research

<http://eprints.mdx.ac.uk>

Novak, Serguei ORCID logoORCID: <https://orcid.org/0000-0001-7929-7641> (1989) Asymptotic expansions in the problem of the longest head run for Markov chain with two states. Trudy Inst. Math. (Novosibirsk), 13 . pp. 136-147. [Article]

UNSPECIFIED

This version is available at: <https://eprints.mdx.ac.uk/20710/>

Copyright:

Middlesex University Research Repository makes the University's research available electronically.

Copyright and moral rights to this work are retained by the author and/or other copyright owners unless otherwise stated. The work is supplied on the understanding that any use for commercial gain is strictly forbidden. A copy may be downloaded for personal, non-commercial, research or study without prior permission and without charge.

Works, including theses and research projects, may not be reproduced in any format or medium, or extensive quotations taken from them, or their content changed in any way, without first obtaining permission in writing from the copyright holder(s). They may not be sold or exploited commercially in any format or medium without the prior written permission of the copyright holder(s).

Full bibliographic details must be given when referring to, or quoting from full items including the author's name, the title of the work, publication details where relevant (place, publisher, date), pagination, and for theses or dissertations the awarding institution, the degree type awarded, and the date of the award.

If you believe that any material held in the repository infringes copyright law, please contact the Repository Team at Middlesex University via the following email address:

eprints@mdx.ac.uk

The item will be removed from the repository while any claim is being investigated.

See also repository copyright: re-use policy: <http://eprints.mdx.ac.uk/policies.html#copy>

ASYMPTOTTIC EXPANSIONS IN THE PROBLEM OF THE LENGTH OF THE LONGEST HEAD-RUN FOR MARKOV CHAIN WITH TWO STATES

S. YU. NOVAK

(abridged version)

Let $\{\xi_i, i \geq 0\}$ be a homogeneous Markov chain with states $\{0;1\}$, transition probabilities $p_{11}=\alpha$, $p_{00}=\beta$, $0 < \alpha < 1$, $\beta < 1$, and initial distribution $P(\xi_0=1)=p$. We set

$$\eta_n = \max\{k \leq n: \max_{0 \leq i \leq n-k} 1\{\xi_{i+1}=\dots=\xi_{i+k}=1\} = 1\} \quad (0)$$

Random variable η_n is known in literature as the length of the longest head-run.

V.L.Goncharov [1] proved that in the case of Bernoulli scheme we have: for any $j \in \mathbb{Z}$

$$P(\eta_n - l \log n < j) = \exp(-(1-\alpha)\alpha^{j-(l \log n)}) + o(1) \quad (n \rightarrow \infty),$$

where \log is to base $1/\alpha$, $[x]$ is the integer part of x , $\{x\} = x - [x]$.

Analogous results for more general situations were obtained in [2-7]. Assertions of LIL type were found in [4,5,8-13]. Moivre [14] seems to be the first who suggested to study the distribution of the length of the longest head-run.

The purpose of this article is to find asymptotic expansions in the limit theorem for the distribution of r.v.

η_n .

§ 1. Main theorem.

Let $\gamma = (1-\alpha)(1-\beta)/\alpha(2-\alpha-\beta)$ and

$$Y_i(k, \phi) = \quad (1.1)$$

$$\phi^i \sum_{j=0}^i \sum_{d=0}^j T_{j-d} \sum_{\mu=0}^d Q_{\mu,d} \sum_{\nu=0}^{i-j} h(\nu, i-j)(\nu!)^{-1} \sum_{\lambda=0}^{\nu} C_{\nu}^{\lambda} (-1)^{d+\mu+\lambda}$$

$$\phi^{\mu+\lambda} (j+\mu)^{(\nu-\lambda)} - (d+\mu)(j+\mu-1)^{(\nu-\lambda)} \phi^{-1} \quad (i \geq 0)$$

where

$$\begin{aligned} i^{(d)} &= i(i-1)\dots(i-d+1) \\ &\quad (d \geq 1), \\ i^{(0)} &= 1, \quad i^{(-d)} = 0 \end{aligned} \quad (1.2)$$

functions T , Q , h are defined by formulae (2.7), (2.16), (2.12).

Note that $Y_1(k, \phi)$, as a function of the first argument, is a polynomial of degree i ; it is a polynomial of degree $2i$ as a function of the second argument.

Theorem 1. For any $m \geq 1$ there exists constant $C_m = C(m, \alpha, \beta, \rho)$ such that for $n > C_m$ there holds

$$\begin{aligned} \sup_{-\infty < j < +\infty} |P(\eta_n - \lfloor \log n \rfloor < j) - e^{-\phi_{n,j}} \sum_{i=0}^{m-1} n^{-i} Y_1(k_{n,j}, \phi_{n,j})| \\ \leq C_m (n^{-1} \ln n)^m \end{aligned} \quad (1.3)$$

where $k_{n,j} = j + \lfloor \log n \rfloor$, $\phi_{n,j} = \gamma \alpha^{j - \lfloor \log n \rfloor}$.

Corollary. For $n \rightarrow \infty$ we have

$$\sup_{-\infty < j < +\infty} |P(\eta_n - \lfloor \log n \rfloor < j) -$$

$$- e^{-\phi_{n,j}} (1 + \phi_{n,j} (1 - \phi_{n,j}) n^{-1} \log n) | = O(n^{-1}) \quad (1.4)$$

Note that the first and the second terms of the expansion both do not depend on the initial distribution of the chain.

§ 2. Some auxiliary results

In the sequel letters C, c (with indexes or without) denote constants which depend on m and chain parameters only.

Theorem 2. *There exist constants $q < 1$ and $C < \infty$ such that*

$$\sup_{k > C} | \mathbb{P}(\eta_n < k) - A(t_0) t_0^{-n-1} | \leq C q^n \quad (2.1)$$

where

$$A(t) = -V(t)/U'(t) ,$$

$$V(t) \equiv V(t, k) = 1 - (\alpha + \beta - 1)t -$$

$$- (p\alpha + (1-p)(1-\beta))\alpha^{k-1}t^k + p(\alpha + \beta - 1)\alpha^k t^{k+1} ,$$

$$U(t) \equiv U(t, k) = W(t) + (1-\alpha)(1-\beta)\alpha^{k-1}t^{k+1} ,$$

$$W(t) = (1-t)(1-(\alpha + \beta - 1)t) ,$$

$t_0 \equiv t_0(k)$ is a root of $U(t, k)$ with minimal modulus.

In the case of Bernoulli $B(\alpha)$ scheme we have $q = \alpha$ and $C = (2 + \alpha(1 + \alpha)) / ((1 - \alpha)(1 - \alpha^2))$.

Lemma 1. *For $k \geq 1$ we have*

$$F(k, t) \equiv \sum_{n=0}^{\infty} \mathbb{P}(\eta_n < k) t^n = V(t)/U(t) \quad (2.2)$$

where $\eta_0 \equiv 0$.

Denote $\kappa = (\alpha + \beta - 1) / (2 - \alpha - \beta)$, $\delta = 1 - p / \gamma - (1 - p) / (1 - \alpha)$, $\rho = (\alpha + \beta - 1) / ((1 - \alpha)(1 - \beta))$, $H_i \equiv 0$ ($i < 0$),

$$H_i \equiv H_i(k) = \quad (2.6)$$

$$= 2^{-i} \sum_{j=0}^{[i/2]} C_{i+1}^{i-2j} (k+2\alpha)^{i-2j} \left[(k+2\alpha)^2 - 4(k-1)\alpha \right]^j \quad (i \geq 0)$$

We put

$$T_i \equiv T_i(k) = \sum_{j=0}^3 q_j H_{i-j} \quad , \quad (2.7)$$

where $q_0=1$, $q_1=\delta-\alpha$, $q_2=\rho\alpha\rho-\alpha\delta$, $q_3=-\alpha\rho\alpha\rho$.

Lemma 2. For all k large enough we have

$$A(1+u) = \sum_{i=0}^{\infty} T_i u^i \quad (2.8)$$

where $u \equiv u(k) = t_0(k)-1$.

Note that

$$| H_i(k) | \leq (k+2|\alpha|)^i \quad , \quad | T_i(k) | \leq Ck^i \quad (2.9)$$

We define polynomials $P_i(\cdot)$ by the equalities $P_0 = 0$,

$$P_m \equiv P_m(k) = \sum_{j=1}^m G_j(k) b_{m-j,j}(k) \quad (k \geq m \geq 1) \quad ,$$

where $G_j(k) = \sum_{i=0}^j C_{k+1}^i \alpha^{j-i}$ and

$$b_{l,j} \equiv b_{l,j}(k) = \sum_{i_1+\dots+i_j=l} P_{i_1} \dots P_{i_j} \quad (l \geq 0 \quad , \quad j \geq 1)$$

Let $v \equiv v(k) = \gamma\alpha^k$.

Lemma 3. For any $m \geq 1$ there exist constants c_m , k_m such that for $k \geq k_m$ we have

$$\left| u/v - \sum_{i=0}^{m-1} P_i v^i \right| \leq c_m (kv)^m \quad (2.10)$$

We introduce functions $\{\tilde{P}_i, i \geq 0\}$ by the equalities

$$\begin{aligned} \tilde{P}_i &= P_i \quad (0 \leq i \leq m) \quad , \\ \tilde{P}_i v^m &= u/v - \sum_{i=0}^{m-1} P_i v^i \end{aligned}$$

We put also

$$b_{l,j,m} = \sum_{\substack{i_1 + \dots + i_j = l \\ \max i_r \leq m}} \tilde{P}_{i_1} \tilde{P}_{i_2} \dots \tilde{P}_{i_j}$$

Note that

$$b_{l,j} = \sum_{\nu=1}^l j^{(\nu)} h(\nu, l) / \nu! \quad (l \geq 1, j \geq 1) \quad (2.11)$$

where $h(\nu, l) \equiv h(\nu, l, k)$ is a polynomial (as function of k) defined by the equalities $h(0, 0) = 1$, $h(0, l) = 0$ ($l \geq 1$),

$$h(\nu, l) \equiv h(\nu, l, k) = \sum_{1 \leq M \leq \nu'} \sum_{(y, z) \in A(\nu, l, M)} (\nu! / z!) \cdot \{P_{y_1}(k)\}^{z_1} \dots \{P_{y_M}(k)\}^{z_M} \quad (l \geq \nu \geq 1) \quad (2.12)$$

Here $\nu' = \min\{\nu; \sqrt{2l}\}$; $y = \{y_1, \dots, y_M\}$; $z = \{z_1, \dots, z_M\}$; $z! = z_1! \dots z_M!$;

$$A(\nu, l, M) = \left\{ (y, z): 1 \leq y_1 < \dots < y_M; \min_i z_i \geq 1; \sum_{i=1}^M z_i = \nu; \sum_{i=1}^M y_i z_i = l \right\}$$

Similarly

$$b_{l,j,m} = \sum_{\nu \geq l/m}^l j^{(\nu)} h_m(\nu, l) / \nu! \quad (2.13)$$

where $h_m(0, 0) = 1$, definition of $h_m(\nu, l)$ differs from that one of $h(\nu, l)$ by using \tilde{P}_i instead of P_i and $A(\nu, l, M, m)$ instead of $A(\nu, l, M)$, where

$$A(\nu, l, M, m) = \left\{ (y, z) \in A(\nu, l, M) : \max_{1 \leq i \leq M} y_i \leq m \right\}$$

Note that $h_m(\nu, l) = h(\nu, l)$ as $l < m$ and

$$|h_m(\nu, l, k)| \leq 2^m m^\nu (c_m k)^l \quad (2.14)$$

$$|b_{l,j,m}(k)| \leq 2^{m(m+1)} j^j (c_m k)^l \quad (2.15)$$

Lemma 4. Let $S_d(i) = \sum_{r=0}^i r^{(d)}$. Then for $d \geq 0$ we have

$$S_d(i) = (i+1)^{(d+1)} / (d+1) = i^{(d)} + i^{(d+1)} / (d+1)$$

Corollary.

$$S_d(i) = d \sum_{j=1}^i S_{d-1}(j-1) \quad (d \geq 1)$$

$$(i+1)S_d(i-1) = (d+2)(d+1)^{-1} S_{d+1}(i) \quad (i \geq 1)$$

$$S_d(i+1) = S_d(i) + dS_{d-1}(i) \quad (d \geq 1)$$

Lemma 5. Let coefficients $r_j(i)$ be defined by the equality

$$(n+1) \cdot \dots \cdot (n+i) \equiv \sum_{j=0}^i r_j(i) n^{i-j},$$

and let

$$\tilde{Q}_{0,d} = 1, \quad \tilde{Q}_{j,d} = \sum_{1 \leq l_1 < l_1+1 < \dots < l_j < d+j} (l_1 l_2 \dots l_j)^{-1}$$

for $1 \leq j < d$. If $d \geq 1$ then we have

$$r_d(i) = \sum_{j=0}^{d-1} \tilde{Q}_{j,d} S_{j+d}(i)$$

There follows from lemmas 4,5 that

$$r_d(i) = \sum_{j=0}^d Q_{j,d} (i+1)^{(j+d)} \quad (d \geq 0) \quad (2.16)$$

where $Q_{0,0} = 1$, $Q_{0,d} = 0$ ($d \geq 1$), $Q_{j,d} = (j+d)^{-1} \tilde{Q}_{j-1,d}$ ($1 \leq j \leq d$).

Lemma 6. Let $a, v \in \mathbb{Z}$; $v \geq 0$. Then

$$i^{(v)} = \sum_{\lambda=0}^v C_v^\lambda a^{(v-\lambda)} (i-a)^{(\lambda)} \quad (2.17)$$

We define $Y_{i,m} = Y_{i,m}(k, \phi)$ by using $h_m(\nu, l)$ instead of $h(\nu, l)$ in formula (1.1). In the sequel $\phi \equiv n\nu$.

Lemma 7. For all k large enough we have

$$A(t_0) t_0^{-n-1} = e^{-\phi} \sum_{i=0}^{\infty} n^{-i} y_{i,m} \quad (3.1)$$

Lemma 8. Let $\psi = \max(1; \phi)$. Then

$$|y_{i,m}| \leq (c\psi^2 \ln n)^i \quad (3.6)$$

Let $k(n) = \log n - \log \ln n^m$ (\log is to base $1/\alpha$).

Lemma 9. If $m > 1$, then for all n large enough we have

$$\sup_{k \in \mathbb{Z}} \left| P(\eta_n < k) - e^{-\phi} \sum_{i=0}^{m-1} n^{-i} y_i \right| \leq Cq^n + \quad (3.8)$$

$$+ 2 \sup_{k \leq k(n)} e^{-\phi} \sum_{i=0}^{m-1} n^{-i} |y_i| + \sup_{k \geq k(n)} e^{-\phi} \sum_{i=m}^{\infty} n^{-i} |y_{i,m}|,$$

where $q < 1$.

Let

$$\tilde{\eta}_n = \max\{k \leq n: \max_{0 \leq i \leq n-k} 1\{\xi_i = \dots = \xi_{i+k-1} = 1\} = 1\}$$

It is easy to see that assertion (1.3) holds if we define Y_i using \tilde{T}_i instead of T_i , where $\tilde{T}_i = \sum_{j=0}^3 \tilde{q}_j H_{i-j}$, $\tilde{q}_0 = 1$, $\tilde{q}_1 = 1 - \alpha + \beta - p/(1-\alpha)(1-\beta)$, $\tilde{q}_2 = (1-\alpha)\rho - \alpha + \alpha\hat{p}/(1-\alpha)(1-\beta)$, $\tilde{q}_3 = -\alpha p$, $\hat{p} = \alpha(p+\beta-1)/(1-\alpha)(1-\beta)$.

§ 4. Remark on the rate of convergence.

Let $\{X_n, n \geq 1\}$ be a Markov chain with state space $S = \{0, 1, \dots, m\}$, transition probabilities p_{ij} and initial distribution \bar{p} . We define r.v. η_n by equality (0), where $\xi_i = 1\{X_i \in A\}$, $A = \{1, \dots, m\}$.

Let λ be a maximal eigenvalue of the matrix $U = \|p_{ij}\|_{i,j \in A}$. We introduce r.v. ζ with distribution

$$P(\zeta=1) = p_{00}, \quad P(\zeta=i) = \bar{p}_{0A} U^{i-2} \bar{p}_{Ao} \quad (i \geq 2),$$

where $\bar{p}_{0A} = \|p_{0j}\|_{j \in A}$, $\bar{p}_{Ao} = \|p_{i0}\|_{i \in A}$. We suppose that there is only one class C of essential states, which has no cyclic subclasses; $A \cap C \neq \emptyset$; $0 < \lambda < 1$; corresponding right eigenvector \bar{z} of matrix U is positive: $z_j > 0$ ($1 \leq j \leq m$).

Theorem 3. Let $\alpha(k) = P(\zeta > k)$ and

$$\Delta(n, k) = |P(\eta_n < k) - \exp(-n\alpha(k)) M(\zeta)| \quad (4.1)$$

§1

Then $\sup_{1 \leq k \leq n} \Delta(n, k) = O(n^{-1} \ln n)$ as $n \rightarrow \infty$.

Let τ_i be the i -th zero in the sequence $\{X_n, n \geq 1\}$ and let $\zeta_i = \tau_i - \tau_{i-1}$. Then

$$\eta_n = \max \{ n - \tau_{\nu(n)}; \max_{1 \leq i \leq \nu(n)} \zeta_i - 1 \} \quad (4.2)$$

where $\nu(n) = \max\{i: \tau_i \leq n\}$. The proof is based on the fact that $P(\eta_n < k) \approx M(1 - \alpha(k))^{\nu(n, k)}$, where $\nu(n, k) = \max\{r: \sum_{j=1}^r \zeta_j^{(k)} \leq n\}$, r.v.'s $\zeta_j^{(k)}$ are independent and have the distribution $P(\zeta_j^{(k)} = i) = P(\zeta_j = i | \zeta_j \leq k)$

REFERENCES

1. Goncharov V.L. *From the domain of combinatorics* // Izv. Akad. Nauk SSSR, Ser. Mat. - 1944. - v.8, No 1. - 3-48.
2. Földes A. *The limit distribution of the length of the longest head-run* // Trans. 8-th Prague Conf. Inform. Th., Statist. Decis. Func., Rand Processes. - Prague, 1979. - 95-104.
3. Kusolitsch N. *Longest runs in Markov chains* // Probab. Statist. Inference. - 1982. - 223-230.
4. Novak S.Yu. *On the length of the longest head-run in Markov chains* // 4th Intern. Vilnius Conf. Probab. Th. Math. Statist. : Abstracts Commun. - Vilnius, 1985. - v.2. - 267-268.
5. Novak S.Yu. *Time intervals of constant sojourn of a homogeneous Markov chain in a fixed subset of states* // Siberian Math. J. - 1988. - v.29, No. 1. - 100-109.
6. Mory T.F., Szekely G. *Asymptotic independence of "pure head" stopping times* // Statist. Probab. Letters. - 1984. - V.2, No 1. - 5-8.
7. Anisimov V.V., Cherniak A.I. *Limit theorems for certain rare functionals on Markov chains and semi-Markov processes* // Theor. Probab. Math. Statist. - 1983. - v.26. - 1-6.
8. Erdős P., Révész P. *On the length of the longest head-run* // Colloq. Math. Soc. J. Bolyai : Topics inform. theory. - 1975. - V. 16. - 219-228.
9. Samarova S.S. *On the number of time intervals that an ergodic Markov chain is continuously in a fixed state* // Dokl. Akad. Nauk SSSR. - 1981. - v.260, No 1. - 35-40.
10. Variakojis L. *On the maximum length of successful series in a enumerable Markov chain* // Lietuvos Matem. Rinkinyje - 1986. - v.26, No 4. - 616-625.
11. Guibas L.J., Odlyzko A.M. *Long repetitive patterns in random sequences* // Z. Wahrscheinlichkeitstheor. Verw. Geb. - 1980. - B.53. - No 3. - 241-262.

12. Novak S. Yu. *On the length of the longest increasing run*
// 1st Congr. Bernoulli Soc. Math. Statist. Probab. Th. -
Tashkent, 1986. - v. 2. - 766.
13. Grill K. *Erdős-Revesz type bounds for the length of the
longest run from a stationary mixing sequence* // Probab.
Th. Rel. Fields . - 1987. - v. 75, No 1. - 77-85.
14. Moivre A. *Doctrine of chances* // L., 1738.
15. Novak S. Yu. *On the time that a homogeneous Markov chain
is continuously in a finite subset of states* //
Theor. Probab. Appl. - 1986. - v. 31, No 2. - 412-413.
16. Bitzadze A. V. *Theory of analitic functions of complex ar-
gument.* - Moscow: Nauka, 1984.
17. Feller W. *An introduction to probability theory and its
applications.* - J. Wiley & Sons, 1970.
18. Borovkov A. A. *Probability Theory.* - Moscow: Nauka,
1986.
19. Borovkov A. A. *Mathematical Statistics.* - Moscow:
Nauka, 1984.

Electrotechnical Institute
korp. 5 / 521
pr. Marksa 20
Novosibirsk 630092
Russia